

CONCEPTUAL STRUCTURES OF GEOMETRY AMONG PRE-SERVICE MATHEMATICS TEACHERS

Dr. Ajay Kumar Mishra
Professor & Vice Chancellor
Sikkim Skill University
Namthang, Sikkim

ABSTRACT

Geometry holds a central role in the mathematics education due to its special ability in the formation of logical thinking, visualization of spaces, and deductive thinking through proofs. In the case of pre-service mathematics teachers, the conceptual knowledge of geometry is particularly important because the knowledge framework of the teacher determines how mathematical ideas on geometry will be taught and learned by other students. In this paper, a conceptualization of the theoretical framework of geometric knowledge among pre-service mathematics teachers will be explored theoretically. The definitions, axioms, postulates, theorems and proof strategies are central components that are addressed in the context of Euclidean geometry. Classical theorems such as the angle-sum property of triangles, triangle congruence, the Pythagorean theorem and circle theorems are introduced by using symbolic equations and the presentation of proofs. Moreover, the van Hiele model of geometric thinking is also pointed at as one of the significant perspectives of the progression of geometric thinking.

Keywords: Geometry Education; Conceptual Structures; Pre-Service Mathematics Teachers; Euclidean Geometry; Axioms and Theorems.

1. INTRODUCTION

Geometry is a field of mathematics that is one of the oldest and most classical and its origins lie in the pragmatic requirements of the early civilizations. In the past, the development of geometric reasoning was a result of land surveying, building construction, astronomical studying and arrangement of the physical space. The name geometry is based upon the Greek words geo (earth) and metron (measure), and it still indicates that geometry was originally aimed at measuring space. In the course of centuries, geometry grew out of empirical observation to become a rigorous deductive study which acquired a formal form with the work of Euclid in The Elements in which geometry was systematized as an axiomatic system.

Geometry today is much more than the study of shapes, lines, and figures in the education of mathematics. It is a very potent field where students learn to visualize space, abstract thinking, logical argument building, and construct proofs. Geo is unlike arithmetic or algebra, which pay much attention to symbolic calculation, and geometry does allow intuition and logical reasoning to interact. As a case in point, the geometric relationships are often formulated using mathematical laws like the angle-sum law of triangles:

$$\angle A + \angle B + \angle C = 180^\circ$$

These assertions are more than computational outputs but are more based on more fundamental structural truths based on geometrical axioms and logical arguments.

Moreover, geometry forms the main point of entry into formal mathematical proof. Congruence, similarity and transformation are concepts which make learners explain relationships in a systematic way. As an example, the Pythagorean relationship of Euclidean space shows the relation between geometry and the representation of space algebraically:

$$a^2 + b^2 = c^2$$

In this way, geometry will serve as a transition between the visual and the symbolic mathematical reasoning, and students will be able to proceed in their intuitive comprehension to formal abstraction.

Geometry is even more vital in teacher education. The pre-service mathematics teachers would not just memorize geometric facts or use formulas but would also have a strong conceptual foundation of geometry. A conceptual structure is an entwined web of geometric notions, definitions, properties, axioms and deductive connections which enable teachers to reason and explain things in a flexible and significant manner. Out of example, one needs to have knowledge of both a geometric definition and an algebraic representation to comprehend the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

Here, the teacher must cognitively connect the definition of a circle as the collection of all points equidistant from a fixed centre to its coordinate formulation.

2. LITERATURE REVIEW

Erdogan (2017) tested the conceptual frameworks of pre-service mathematics teachers in terms of geometry. The experiment based on the way in which future teachers designed and comprehended the basic geometric ideas including points, lines, angles, and shapes. Erdogan emphasized that most pre-

service teachers had discontinuous conceptual knowledge despite having an exposure to the use of geometric terms. Results have shown that participants tended to use memorized definitions instead of conceptual relationships between the ideas of geometry. The author underlined that poor conceptual structures might have a detrimental impact on the capacity of teachers to describe geometric relationships and help students to reason in terms of proofs. The work by Erdogan helped to appreciate the role of enhancing conceptual grounds in teaching of geometry.

Gambini and Lénárt (2021) explored fundamental geometrical concepts in instruction in the reasoning of in-service and pre-service mathematics teachers. Their study involved the examination of the conceptualizations of teachers engaged in various professional stages of the essential geometric objects and properties. According to the authors, pre-service teachers often had problems with constructing logical connections between definitions, pictorial representation, and inductive logic. Comparatively, in-service teachers showed more coordinated knowledge of geometry because of classroom experience. The research found out that learning in geometry with regard to development of concepts was greatly affected by exposure to reasoning, visualization activities, and formal proof practices. Gambini and Levenart found that teacher education programs should pay more attention to generating profound conceptual understanding of the basic ideas of geometry.

Oflaz et al. (2016) delved into the evidence schemes of pre-service classroom teachers in geometry using a case study approach. The researchers studied the way of tackling the tasks on geometric proofs by three pre-service teachers and the strategies used to reason. The results indicated that the respondents tended to build evidences on observational and diagram-driven premises as opposed to formal deductive reasoning. The authors found that most pre-service teachers had difficulties linking axioms, definitions and theorems in a systematic way in order to prove geometric statements. The research focused on highlighting the fact that insufficient learning of formal proof writing also led to poor deductive reasoning ability. Oflaz et al. proposed that geometry teaching in the future-to-future teachers should put more emphasis on the development of proofs and the justification techniques.

Pirasa (2016) examined the relationship between the competencies of pre-service mathematics teachers in geometric concepts and their connection to life in general. The paper examined the relationship of the abstract geometric concepts to the real life and practical scenarios of the future teachers. Pirasa has said that most pre-service teachers had some difficulties in the process of transferring knowledge of geometry outside the classroom. Participants were able to do regular calculations, however, in reality, they did not conceptually interpret geometry in real world. The study

showed that the skill of connecting the geometric concepts of symmetry, of area, of relationships of space to everyday experiences was a necessary part of significant teaching. Pirasa made the conclusion that the teacher training programs must include the contextual and applied geometry tasks to increase conceptual relevance and grasp.

3. CONCEPTUAL STRUCTURE IN GEOMETRY

Conceptual structure in geometry is the cognitive structure internal to mathematics in which mathematical concepts are organized, linked and comprehended. The conceptual understanding of mathematics has been defined as the ability to remember formulas or recognise shapes in mathematics education, but to understand the relationships, justify the properties and use reasoning in a variety of geometrical situations. A conceptual structure is thus a web of knowledge which allows learners and teachers to go beyond the manipulation of the procedures towards the understanding of the meaning of geometry which learners develop.

A conceptual structure comprises a number of components:

3.1 Definitions as Conceptual Anchors

The foundations of geometrical knowledge take the form of definitions. They give accurate meaning to geometrical objects and relationships. Geometric definitions are precise and definite unlike colloquial language.

To take the example of a circle, it is not a mere round shape, but it is defined formally as:

A circle is the set of all points in a plane that are equidistant from a fixed point called the centre.

This definition can be represented symbolically in coordinate geometry as:

$$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) is the center and r is the radius.

In the case of pre-service teachers, conceptual definition is being aware of the fact that the concept of definition is not a description but the tool on which any further reasoning is based.

3.2 Axioms and Postulates as Foundational Assumptions

Geometry works in axiomatic system. A statement without proof is called an axiom or postulate and it is a fundamental truth on which theorems are based.

As an example, the postulate made by Euclid:

Two points can be joined using a straight line.

This creates the option of logical construction of geometric figures.

In the same way, the parallel postulate makes the foundations of Euclidean geometry:

Consider line l and a point P not on l , there is one line that is through P parallel to l .

These axioms are the structural foundations of the whole system of geometry. The pre-service teachers should learn that axioms are not what are proven, but rather the initial assumptions of a deductive reasoning.

3.3 Theorems as Logical Consequences

Theorems are statements in mathematics which are demonstrated using axioms, definitions, and already known results. Theorems are logical extensions of geometry out of basic truths to complicated relations.

As an example, the angle-sum law of a triangle:

$$\angle A + \angle B + \angle C = 180^\circ$$

is not assumed but derived deductively from Euclidean postulates.

Similarly, the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

is a geometrical correspondence which is due to the constitution of Euclidean space.

Conceptually, the knowledge of theorems implies the perception of their belonging to a logical chain, not the memorization of formulas.

3.4 Proof Strategies and Deductive Reasoning

The focus on proof is one of the most peculiar aspects of geometry. Proof This is a process by which geometric statements are logically justified.

A proof typically follows the structure:

Given $\rightarrow$ Assumptions $\rightarrow$ Logical Steps $\rightarrow$ Conclusion

Proof strategies include:

- direct proof
- contradiction

- construction-based proof
- transformation proof
- coordinate and algebraic proof

Indicatively, the congruence of triangles via SAS only makes sense when one has a unique knowledge that two of the sides and the angle between the sides and triangle will yield a triangle.

$$AB = DE, AC = DF, \angle A = \angle D \Rightarrow \triangle ABC \cong \triangle DEF$$

Pre-service teachers are usually challenged by the concept of proof as it requires them to have understanding of the concepts, sequence of logic and connection of the abstract reasoning and the visual diagrams.

4. FOUNDATIONS OF EUCLIDEAN GEOMETRY

Euclidean geometry is the most traditional and most commonly studied system of geometrical thought, which is the staple of the teaching of mathematics during the last two millennia. Its roots were made systematically available, by the Greek mathematician Euclid, in his master work Elements, in which geometry was not to be conceived as a collection of isolated facts, but as a logically structured system of deductions. The method of Euclid brought geometry to an axiomatic system, starting with a limited number of undefined primitive concepts, an explicit definition and a few axioms that were not proved to be true.

Out of these fundamental premises, a whole body of geometrical knowledge was developed by way of sound logical thinking and demonstrating. Euclidean system is still vital in the comprehension of how mathematical entities like points, lines, planes, angles, and parallelism are derived in a consistent mathematical system. In the case of pre-service mathematics teachers, having a good understanding of these Euclidean foundations is imperative since it gives them a conceptual context into which they can teach geometry in a manner that entails reasoning, development of theorems, and proof-based comprehension instead of memorization.

4.1 Basic Geometric Definitions

Definition 1 (Point)

A point is an undefined primitive term representing an exact location in space with no dimension.

Definition 2 (Line)

A line is a straight one-dimensional figure extending infinitely in both directions.

Definition 3 (Plane)

A plane is a two-dimensional flat surface extending infinitely.

4.2 Euclid's Postulates

Euclid proposed five postulates. The most important are:

1. A straight line can be drawn joining any two points.
2. A finite line can be extended indefinitely.
3. A circle can be drawn with any center and radius.
4. All right angles are equal.
5. (*Parallel Postulate*) Given a line and a point not on it, exactly one line parallel to it can be drawn.

5. KEY THEOREMS AND PROOF STRUCTURES

One of the main differences between geometry and other school mathematics is its focus on the logical deduction and proof. Geometry exposes students and teachers to a very special form of mathematical reasoning, in contrast to arithmetic (which is frequently concerned with computation), or algebra (which is frequently concerned with manipulation of symbols). Evidence is then regarded to be the essence of geometric thought and is the source in which geometry knowledge is justified and systematized.

Evidence is the core of mathematics and another dominant conceptual problem to pre-service teachers.

5.1 Triangle Geometry**Theorem 1: Angle Sum Property of a Triangle****Statement**

The sum of the interior angles of a triangle is 180° .

$$\angle A + \angle B + \angle C = 180^\circ$$

Proof

Let $\triangle ABC$ be a triangle.

Draw a line through point A parallel to BC.

By alternate interior angles:

$$\angle ABC = \angle A_1$$

$$\angle ACB = \angle A_2$$

Now, angles on a straight-line sum to 180° :

$$\angle A_1 + \angle A + \angle A_2 = 180^\circ$$

Substituting:

$$\angle B + \angle A + \angle C = 180^\circ$$

Hence proved.

5.2 Congruence and Conceptual Understanding

Definition (Congruence)

Two triangles are congruent if all corresponding sides and angles are equal.

Theorem 2: SAS Congruence Criterion

If:

$$AB = DE, AC = DF, \angle A = \angle D$$

then:

$$\triangle ABC \cong \triangle DEF$$

Proof (Sketch)

Through rigid motion (rotation/translation), the triangle can be superimposed perfectly on the other triangle since two sides and the angle between them make the triangle unique.

The rest is therefore to be equal.

5.3 Pythagoras Theorem

Theorem 3: Pythagoras Theorem

For a right triangle:

$$a^2 + b^2 = c^2$$

where c is the hypotenuse.

Proof (Area Proof)

Construct squares on each side of the triangle.

Total area of square on hypotenuse equals sum of areas of squares on legs.

$$\text{Area}(c) = \text{Area}(a) + \text{Area}(b)$$

Thus:

$$c^2 = a^2 + b^2$$

Hence proved.

Example Solution

Given:

$$a = 6, b = 8$$

$$c^2 = 6^2 + 8^2 = 36 + 64 = 100$$

$$c = \sqrt{100} = 10$$

6. CIRCLES AND GEOMETRY OF CURVATURE

Definition (Circle)

A circle is the set of all points equidistant from a fixed point (center).

Equation:

$$(x - h)^2 + (y - k)^2 = r^2$$

Theorem 4: Angle in a Semicircle

An angle subtended by a diameter at the circumference is a right angle.

Proof

Let AB be diameter and C any point on circle.

By circle theorem:

$$\angle ACB = 90^\circ$$

Hence proved.

7. CONCEPTUAL CHALLENGES AMONG PRE-SERVICE TEACHERS

There are a number of conceptual problems that pre-service mathematics teachers are likely to face in geometry, and they include working with definitions, theorems, symbolic relationships, and proof-based reasoning. Such difficulties are not just pedagogical but they are also mathematical because geometry demands proper interpretation of axioms, usage of theorems and capability to relate axioms to space and algebraic forms to space.

The most common conceptual difficulties are presented below containing the main formulas, equations, and examples of the theorems.

7.1 Misunderstanding Axioms vs Theorems

A great problem occurs when the teachers cannot differentiate statements that are accepted without evidence and statements that were obtained logically.

Examples of Axioms (Accepted Statements) Through two distinct points, exactly one straight line can be drawn.

$$P \neq Q \Rightarrow \exists! PQ$$

Parallel postulate:

Given a line l and a point $P \notin l$:

$$\exists! m \text{ such that } P \in m \text{ and } m \parallel l$$

Examples of Theorems (Proved Statements)

- Angle sum theorem:

$$\angle A + \angle B + \angle C = 180^\circ$$

- Pythagoras theorem:

$$a^2 + b^2 = c^2$$

Such theorems are mistaken by many pre-service teachers as axioms, and they do not realize the deductive format of proof behind them.

7.2 Weak Proof Construction Skills

Evidence demands the use of chain of theorems and logical processes, yet the use of half-baked logic is common with many teachers.

Example: Proof of Triangle Congruence

SAS Criterion:

If

$$AB = DE, AC = DF, \angle A = \angle D$$

Then:

$$\triangle ABC \cong \triangle DEF$$

Teachers tend to mention congruence randomly and without the logical implication:

$$\text{Given} \Rightarrow \text{Criterion} \Rightarrow \text{Conclusion}$$

Example: Parallel Lines and Alternate Angles

If $l \parallel m$, then:

$$\angle 1 = \angle 2$$

This finding is frequently used by many pre-service teachers without reference to the parallel postulate.

7.3 Reliance on Memorization of Formulas

Most teachers learn geometric formulas by heart as opposed to the conceptual reasoning.

Commonly Memorized Formulas

Area of triangle:

$$A = \frac{1}{2}bh$$

Circumference of circle:

$$C = 2\pi r$$

Area of circle:

$$A = \pi r^2$$

Volume of cylinder:

$$V = \pi r^2 h$$

But teachers do not tend to give an explanation as to why these formulas are geometric.

Example Misconception

Using the formula:

$$A = \pi r^2$$

even without knowing that it is the result of a rearrangement of a circle into a rough rectangle:

$$A \approx (\pi r) \cdot r$$

8. IMPORTANCE OF STRONG CONCEPTUAL STRUCTURES

The application of good conceptual framework in geometry is a necessity to pre-service mathematics teachers since geometry is not merely a formula and shape only, but a system of logic that is grounded in reasoning, deduction, and proof. When the teacher has strong conceptual knowledge, he can transcend procedural teachings and help students reach actual mathematical thought processes. Conceptual structures enable teachers to relate definitions, axioms and theorems into some logical organization that geometry is taught as a significant and logically structured field.

Educators having highly- developed conceptual frameworks have been able to describe geometric relationships in an understandable language, prove theorems by providing proofs, and assist students to comprehend why mathematical properties in geometry are universal. As an example, rather than teaching the angle-sum property as a memorization rule:

$$\angle A + \angle B + \angle C = 180^\circ$$

a conceptually powerful teacher shows how this theorem can arise out of parallel lines axiom and alternate interior angles. This way, geometry is learnt by reasoning and not memorization.

8.1 Meaningful Presentation of Geometry

Educationalists who have conceptual power are able to introduce the world of geometry as which ideas are related and not isolated facts. They assist students in developing a gradually acquired knowledge starting with very specific definitions like:

A circle is a group of all those points which are equally far away as a given center:

$$(x - h)^2 + (y - k)^2 = r^2$$

In such teaching, it is stressed that formulas do not occur randomly but they are geometric meaning direct derivations.

8.2 Development of Students' Logical Reasoning

Geometry is among the best instruments in building reasoning skills as it contains the requirement to provide justification of all conclusions. A teacher that possesses a powerful conceptual framework can guide the students through the process of making the observation to deduction.

As an example, triangles congruency is not measured only, but the basis is the logical measurements like SAS:

If:

$$AB = DE, AC = DF, \angle A = \angle D$$

Then:

$$\triangle ABC \cong \triangle DEF$$

This reinforces the mathematical reasoning capacity of students by argument structure.

9. CONCLUSION

The conceptual structures of geometry among pre-service mathematics teachers play a fundamental role in ensuring effective and meaningful mathematics instruction, as geometry is not merely a collection of shapes, formulas, or isolated facts but an axiomatic and proof-oriented branch of mathematics that demands deep conceptual understanding, logical connections, and strong deductive reasoning skills. For future teachers, possessing well-organized and interconnected geometric knowledge is essential because it forms the foundation for developing students' spatial thinking, problem-solving abilities, and mathematical reasoning. Moreover, the development of strong conceptual structures requires more than procedural competence; incorporating foundational definitions, classical theorems, and systematic proof strategies strengthens the cognitive framework of pre-service teachers, enabling them to interpret geometric ideas with clarity and confidence. Teachers

who understand geometry conceptually are better equipped to explain relationships among concepts, guide learners through reasoning processes, and foster a classroom culture of inquiry and justification.

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